Mastering the Chain Rule

The Chain Rule tells us how to differentiate a function formed by the *composition,* or sequential application, of two simpler functions.

The key to understanding the Chain Rule is to think about a function as a *process* which transforms one number into another. We are used to calling the input number x and the output y, but this is a situation where we must put that convention aside.

Describe the Process

When you look at a look at a function such as $h(x) = \sqrt{x^2 - 4x + 9}$, you need to practice breaking it up into its component steps. What does h do to the input number? At a very fine level of detail, h squares the input number, subtracts four times the input number from that, adds nine, then takes the square root. You should be able to describe the process without using variable names. This is important; we must separate the function from the particular variable names.

For practical use, we don't need all the details: it suffices to say that h "does the function $g(x) = x^2 - 4x + 9$," and then "does the square root function."

The Importance of Names

If we want to write down the square root function, we immediately risk confusion, because the traditional way to write this is $f(x) = \sqrt{x}$. If we use this notation here, we are courting disaster. x has a different meaning here! The number that we'll be taking the square root of is not x!

So let's introduce another variable name, u. Now we can picture the process this way:

$$x \xrightarrow{g} u \xrightarrow{f} y$$

and we can write formulas for our functions that match this notation:

$$\begin{aligned} u &= g(x) = x^2 - 4x + 9 \\ y &= f(u) = \sqrt{u} \end{aligned}$$

Of course, the function f is the square root function, whether we call the input number x, u, β, or anything else. But using different variable names for different stages in the process will help us avoid confusion. When you are dealing with a composition of functions, I strongly recommend that you first rewrite the definitions of the functions, using variable names which represent the different stages in the process.

The Simple Idea Behind the Chain Rule

Remember that the derivative $\frac{dy}{dx}$ measures rate of change. “Rate” means ratio, which means division, and the derivative starts with $\frac{\text{change in } y}{\text{change in } x}$. (The derivative also involves a limit, but let’s focus on the ratio.)

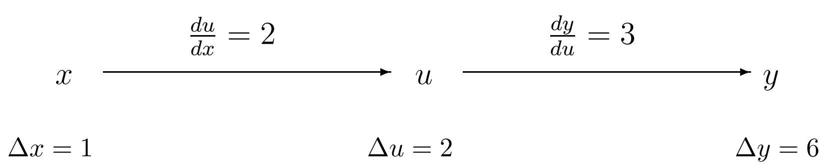
By giving a name to the intermediate number, we gain the ability to write:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx},$$

or, in English

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\text{change in } y}{\text{change in } u} \times \frac{\text{change in } u}{\text{change in } x}.$$

This makes sense. If the derivative $\frac{dy}{du}$ is 3, for instance, this means that y changes 3 units for every unit that u changes. If, at the same time, u is changing 2 units for every unit of change in x, then clearly 1 unit of change in x will lead to 2 units of change in u, which in turn will produce 6 units of change in y. This is summed up in this diagram:



The Tricky Twist to the Chain Rule

So far, everything seems clear: if we want the derivative of a composite function $f \circ g$, we find the derivative of f, and the derivative of g, and we multiply them together. This description is correct, as far as it goes, but it doesn’t tell the whole story. The tricky part involves *where we evaluate the derivatives.*

Let’s use our example function $h(x) = \sqrt{x^2 - 4x + 9}$, and calculate the derivative at a point. Let’s find $h'(3)$. Actually, before we do that, let’s evaluate $h(3)$, using the two steps g and f.

$$g(3) = 3^2 - 4(3) + 9 = 6$$

and then

$$f(6) = \sqrt{6}.$$

Expressed in a diagram, the situation is:

$$3 \xrightarrow{g} 6 \xrightarrow{f} \sqrt{6}.$$

Now, to find the derivative of our original function, we need to answer two questions:

1. What is the derivative of each component function?
2. Where do we evaluate each of those individual derivatives?

First, the individual derivatives are

$$g'(x) = 2x - 4$$
$$f'(u) = \frac{1}{2\sqrt{u}}$$

To find the derivative $x = 3$, we plug 3 into the expression for $g'(x)$, *and we plug 6 into the expression for* $f'(u)$.

$$\left.\frac{du}{dx}\right|_{x=3} = g'(3) = 2(3) - 4 = 2$$

$$\left.\frac{dy}{du}\right|_{u=6} = f'(6) = \frac{1}{2\sqrt{6}}.$$

In our shorthand notation, we have $\frac{du}{dx} = 2$, $\frac{dy}{du} = \frac{1}{2\sqrt{6}}$, and thus

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2 \times \frac{1}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

Notice, though, that this shorthand does not record where the derivative is evaluated. So although the shorthand nicely expresses the simple idea behind the Chain Rule, it is not sufficient for doing calculations.

Notice that we plugged 3 into the derivative of g, but we plugged 6 into the derivative of f. That's because in this situation, 6 was the input to f, so we are interested in the derivative of f *when its input is 6.*

The most common error in using the Chain Rule is to put the original x value into both derivatives. We can't do this. The number substituted into each derivative must be the number that is relevant at that stage of the multi-step process.

A Numerical Example

Let's look at a numerical example, still using $h(x) = \sqrt{x^2 - 4x + 9}$. When $x = 3$, we have this picture:

$$3 \xrightarrow{g} 6 \xrightarrow{f} \sqrt{6} \approx 2.4495$$

Now let's change x a small amount, say by 0.1, and observe how much change this causes in u and in y. We do the calculation, and get

$$3.1 \xrightarrow{g} 6.21 \xrightarrow{f} \sqrt{6.21} \approx 2.9420.$$

Here, the ratio $\frac{\text{change in } u}{\text{change in } x}$ is $\frac{.21}{.1} = 2.1$, which is close to the derivative of g when the input is 3. $g'(3) = 2(3) - 4 = 2$. (If the change in x were smaller, the ratio would be closer to $g'(3)$. In fact, the original definition of the derivative is "the number that this ratio gets closer and closer to as the size of the change gets very small.")

Similarly, we have $\frac{\text{change in } y}{\text{change in } u} \approx \frac{.0453}{.21} \approx .207$. This should be close to the derivative of f, *as long as we evaluate that derivative at the correct point!* The input to f is 6, so we calculate $f'(6) = \frac{1}{2\sqrt{6}} \approx .204$, which is indeed very close to $\frac{\text{change in } y}{\text{change in } u}$.

Where the Chain Rule Formula Comes From

Let's stick with our example function $h(x) = \sqrt{x^2 - 4x + 9}$, and think about its derivative at an arbitrary x value. We saw above that when $x = 3$, we have to compute the intermediate value $u = g(3) = 6$, and use this u value in the derivative of f. So we have

$$h'(3) = g'(3) \times f'(6),$$

or

$$h'(3) = g'(3) \times f'(g(3)).$$

This is the general rule, applicable to any x and any function h made by composing functions called g and f. It is customary to write the factors in the other order, and to omit the times sign:

The Chain Rule

If $h = f \circ g$, which is also written $h(x) = f(g(x))$, then the derivative of h is

$$\begin{aligned} h'(x) &= f'(u)g'(x) \\ &= f'(g(x))g'(x), \end{aligned}$$

where $u = g(x)$.

The key thing to remember is that whatever the original input to a function was, that same input is used for its derivative. The first function, g, had input x, and that's why we see $g'(x)$ as part of the derivative. The second function, f, had input $g(x)$, and that's why we see $f'(g(x))$. This is a very useful fact to keep in mind. If we are differentiating $h(x) = 3\sin(5x^2 - 3x + 2)$, for instance, we know that the derivative will involve the cosine function, *and* we know that it will be the cosine of $5x^2 - 3x + 2$. This was the input to the sine function, so it will be the input to the derivative of the sine function.

Two Traps to Avoid

Most problems in applying the Chain rule involve one of these two errors.

<table>
<tr><th colspan="4">Common Errors in Differentiating</th></tr>
<tr><th>The Error in Words</th><th>The Error in Symbols</th><th>An Example</th><th>Correct Calculation</th></tr>
<tr><td>Evaluating both derivatives at the original x value.</td><td>$f'(x)g'(x)$</td><td>$\frac{d}{dx}\sin(x^2+4)$
$\neq \cos(x) \times 2x$</td><td rowspan="2">$\frac{d}{dx}\sin(x^2+4)$
$= \cos(x^2+4) \times 2x$</td></tr>
<tr><td>Plugging one derivative into the other</td><td>$f'(g'(x))$</td><td>$\frac{d}{dx}\sin(x^2+4)$
$\neq \cos(2x)$</td></tr>
</table>

To guard against the second error, remind yourself that we differentiate only one function at a time. While we are differentiating the sine function, the function $x^2 + 4$ just hangs around waiting for its turn.

Compositions of More Than Two Functions

Finding derivatives is what computer scientists call a *recursive process.* This means that in the process of finding the derivative, we have to find the derivative of something else. Fortunately, the functions we need to differentiate get simpler at each stage.

To find the derivative, we always begin with the *last* function done. Use the process way of thinking. Imagine evaluating the function for a particular number x, and identify the *last* thing you do. That's where you start the differentiation.

The recursion is expressed in the phrase "times the derivative of the inside." This mantra reminds us that after we've written down the derivative of the last step in the process, we look at the step just before that one, and apply the whole differentiation process to that previous step. We follow the chain of steps backward until we reach the original input number, and then stop.

Once you become comfortable with the Chain Rule, you will not always assign variable names to the intermediate stages, as we did with u above. Instead, you will use the thinking process illustrated in the next example.

Example: Differentiate the function $f(x) = \cos(\sqrt{x^3 - 5})$.

First, we think about the process that f performs on its input number. Lumping the first two steps together, we'll think of this as:

1. cube the input and subtract 5
2. take the square root
3. apply the cosine function

In writing down the derivative, we'll differentiate each of these three steps, in reverse order.

Differentiating $\cos(\sqrt{x^3-5})$		
What we write	What Function We're Differentiating	What we think
$f'(x) = -\sin(\sqrt{x^3-5}) \times \cdots$	cosine	The derivative of cos(*something*) is $-\sin$(*something*), and the *something* stays the same. The dots show where we're going to multiply by the derivative of the inside. Here, the inside is $\sqrt{x^3-5}$.
$f'(x) = -\sin(\sqrt{x^3-5}) \times \dfrac{1}{2\sqrt{x^3-5}} \times \cdots$	square root	The derivative of $\sqrt{something}$ is $\frac{1}{2\sqrt{something}}$, and the *something* stays the same. The dots show where we're going to multiply by the derivative of the inside. Here, the inside is x^3-5.
$f'(x) = -\sin(\sqrt{x^3-5}) \times \dfrac{1}{2\sqrt{x^3-5}} \times 3x^2$	x^3-5	We've multiplied by the derivative of x^3-5, which requires no further dissection, so we're done.
$f'(x) = \dfrac{-3x^2\sin(\sqrt{x^3-5})}{2\sqrt{x^3-5}}$		We combine the factors.

Parting Shot

The key to mastering the Chain Rule is to see a function as a process.